UNIVERSITÀ DI PAVIA

Università della Svizzera italiana

Optimal Power Flow problem: a study on Jabr relaxation

The HEXAGON Workshop on power grids

Ambrogio Maria Bernardelli   Stefano Gualandi   Gabor Riccardi

19 June 2024

Overview

Problem definition

The problem

Let us take a network modeled as a graph $(\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ represents the set of buses and $\mathcal{L}$ represents the set of lines. For every bus k we have a (possibly empty) set of generators $\mathcal{G}(k)$ located at bus k. The problem consists of meeting the energy demand at every bus, and doing so with the lowest possible energy generation cost.

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More precisely, we have the following variables:

- for each bus k we have a complex voltage $V_k = |V_k|e^{j\delta_k}$;
- for each branch km we have two variables S_{km} and S_{mk}, the complex power injected into the branch at k and at m, respectively;
- for each generator g there is power generation $P_g^G + jQ_g^G$.

These variables are subjected to five classes of constraints.

Polar coordinates formulation

$$\inf_{\substack{P_g^G, Q_g^G, \delta_k, \\ |V_k|, S_{km}}} \sum_{g \in \mathcal{G}} F_g(P_g^G, Q_g^G) \tag{1a}$$

s.t.

AC power flow laws:

$$S_{km} = (G_{kk} - jB_{kk})|V_k|^2 + (G_{km} - jB_{km})|V_k||V_m| \cdot (\cos(\theta_{km}) + j\sin(\theta_{km})) \qquad \forall km \in \mathcal{L}, \tag{1b}$$

Flow balance constraints:

$$\sum_{km \in L} S_{km} + P_k^L + jQ_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + j \sum_{g \in \mathcal{G}(k)} Q_g^G \qquad \forall k \in \mathcal{B}, \tag{1c}$$

Branch limits, generator limits, voltage bounds:

$$|S_{km}|^2 \leq U_{km} \qquad \forall km \in \mathcal{L}, \tag{1d}$$

$$P_g^{\min} \leq P_g^G \leq P_g^{\max},\ Q_g^{\min} \leq Q_g^G \leq Q_g^{\max} \qquad \forall g \in \mathcal{G}, \tag{1e}$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max} \qquad \forall k \in \mathcal{B}, \tag{1f}$$

$$\theta_{km}^{\min} \leq \theta_{km} \leq \theta_{km}^{\max} \qquad \forall km \in \mathcal{L}. \tag{1g}$$

Variable substitution

One can introduce auxiliary variables to tackle the problem of having sine and cosine functions:

$$
\begin{aligned}
c_{km} &= |V_k||V_m| \cdot \cos(\theta_{km}) && \forall km \in \mathcal{L}, \\
s_{km} &= |V_k||V_m| \cdot \sin(\theta_{km}) && \forall km \in \mathcal{L}, \\
c_{kk} &= |V_k|^2 && \forall k \in \mathcal{B}.
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\end{aligned}
$$

Substituting such variables in the model without adding their definitions gives us a first relaxed model.

Note that by doing so we manage to remove sine and cosine functions but we also lose crucial relations between the new variables.

A first relaxed model

$$\inf_{\substack{P_g^G, Q_g^G, c_{km}, \\ s_{km}, S_{km}, P_{km}, Q_{km}}} \quad F(x) := \sum_{g \in \mathcal{G}} F_g(P_g^G) \tag{2a}$$

Subject to:

$$P_{km} = G_{kk} c_{kk} + G_{km} c_{km} + B_{km} s_{km} \qquad \forall km \in \mathcal{L}, \tag{2b}$$

$$Q_{km} = -B_{kk} c_{kk} - B_{km} c_{km} + G_{km} s_{km} \qquad \forall km \in \mathcal{L}, \tag{2c}$$

$$S_{km} = P_{km} + jQ_{km} \qquad \forall km \in \mathcal{L}, \tag{2d}$$

$$\sum_{km \in L} S_{km} + P_k^L + jQ_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + j \sum_{g \in \mathcal{G}(k)} Q_g^G \qquad \forall k \in \mathcal{B}, \tag{2e}$$

$$P_{km}^2 + Q_{km}^2 \leq U_{km} \qquad \forall km \in \mathcal{L}, \tag{2f}$$

$$V_k^{\min^2} \leq c_{kk} \leq V_k^{\max^2} \qquad \forall k \in \mathcal{B}, \tag{2g}$$

$$P_g^{\min} \leq P_g^G \leq P_g^{\max}, \; Q_g^{\min} \leq Q_g^G \leq Q_g^{\max} \qquad \forall g \in \mathcal{G}, \tag{2h}$$

$$c_{kk} \geq 0 \qquad \forall k \in \mathcal{B}, \tag{2i}$$

$$V_k^{\max} V_m^{\max} \geq c_{km} \geq 0 \qquad \forall km \in \mathcal{L}, \tag{2j}$$

$$- V_k^{\max} V_m^{\max} \leq s_{km} \leq V_k^{\max} V_m^{\max} \qquad \forall km \in \mathcal{L}, \tag{2k}$$

$$c_{km} = c_{mk}, \; s_{km} = -s_{mk} \qquad \forall km \in \mathcal{L}. \tag{2l}$$

Jabr (I)

Equality

To link the c and s variables we make use of the following equality:

$$c_{km}^2 + s_{mk}^2 = c_{kk} c_{mm} \quad \forall km \in \mathcal{L}. \tag{3}$$

We will denote by Jabr equality ACOPF relaxation the model (2) together with constraints (3).

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These nonconvex couplings constraints can be relaxed as follows.

Inequality

$$c_{km}^2 + s_{mk}^2 \leq c_{kk}c_{mm} \quad \forall km \in \mathcal{L}. \tag{4}$$

Jabr (II)

Note that inequality

$$c_{km}^2 + s_{mk}^2 \leq c_{kk} c_{mm}$$

can be rewritten as

$$c_{km}^2 + s_{mk}^2 + \left(\frac{c_{mm} - c_{kk}}{2} \right)^2 \leq \left(\frac{c_{mm} + c_{kk}}{2} \right)^2,$$

which represents a rotated SOCP cone in $\mathbb{R}^4$. Note also that the cone (4) is the convex hull of (3).

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Trees and cycles

Trees

Can we take advantage of the network structure to ensure exactness of a relaxed model?

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Lemma 1.

If $(\mathcal{B}, \mathcal{L})$ is a multisource radial network, then the Jabr equality ACOPF relaxation is exact [Jab06].

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More specifically, does a multisource radial network require other constraints other than the Jabr equality?

Lemma 1.

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Why do we need a tree structure for the exactness of the model?

Loop constraints

Definition 1 (Loop constraint).

Given a cycle $\mathcal{C}$ on nodes $\{k_1, \dots, k_n\}$, we define the loop constraint on $\mathcal{C}$ as the following

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A| = 2j}} (-1)^j \prod_{h \in A} s_{k_h k_{h+1}} \prod_{l \in A^c} c_{k_l k_{l+1}} = \prod_{i=1}^{n} c_{k_i k_i}, \tag{5}$$

with $A^c := [n] \setminus A$.

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with $A^c := [n] \setminus A$.

Lemma 2.

The Jabr equality ACOPF relaxation together with the additional loop constraint (5) written for every cycle of $(\mathcal{B}, \mathcal{L})$ is exact.

Constraint redundancy

Definition 2 (Cycle space).

The (binary) cycle space of an undirected graph is the set of its even-degree subgraphs.

Definition 3 (Cycle basis).

A cycle basis of an undirected graph is a set of simple cycles that forms a basis of the cycle space of the graph.

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Lemma 3.

It is sufficient to write (5) for every cycle in a cycle basis of $(\mathcal{B}, \mathcal{L})$.

Linearizations

3-cycles and 4-cycles

We first focus on short cycles, namely, cycles made up of 3 or 4 nodes [KDS16]. Note that, in this case, the polynomials constituiting the loop constraints are cubic polynomials and quartic polynomials, respectively.

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In this particular case, it is possible to reduce the degree of the polynomials thanks to the following result.

Lemma 4.

Given a cycle of length 3 or 4, if the Jabr equality is satisifed on all the branches of the cycle, the loop constraint (5) can be replaced exactly by two bilinear constraints.

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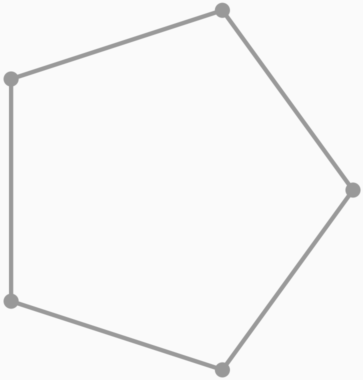
What about larger cycles?

Larger cycles



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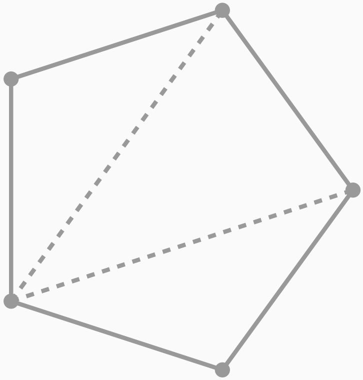

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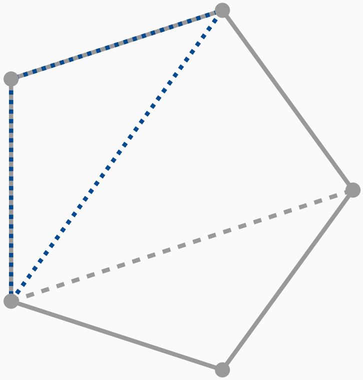

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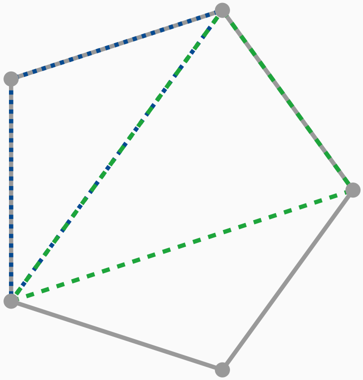

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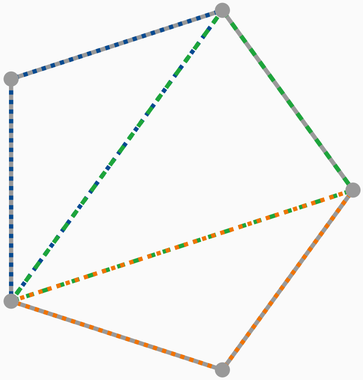

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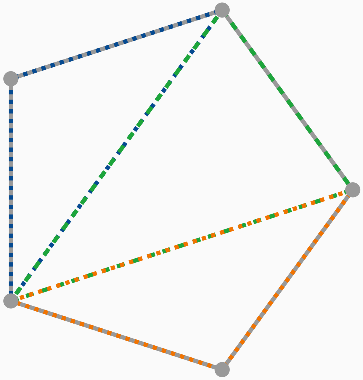

The idea

Decomposing bigger cycles into smaller cycles by creating artificial branches.

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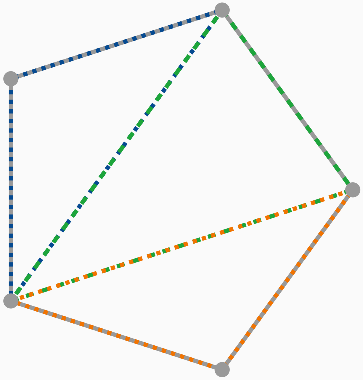

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Two types of decomposition

Cycles can either be decomposed into 3-cycles by adding branches $(1, i)$, for $i = 3, \ldots, n$, or into 4-cycles by adding branches $(1, 2i)$, for $i = 2, \ldots, (n-2)/2$. Note that if n is odd, one 3-cycle needs to be added by creating the artificial branch $(1, n-1)$.

Excursus: multilinear optimization (I)

Multilinear problem

$$\min \sum_{I \in \mathcal{I}_0} c_I^0 \prod_{v \in I} x_v \tag{6a}$$

$$\text{s.t.} \sum_{I \in \mathcal{I}_j} c_I^j \prod_{v \in I} x_v \leq b_j \qquad \forall j \in \{1, \dots, m\}, \tag{6b}$$

$$x_v \in [l_v, u_v] \qquad \forall v \in V, \tag{6c}$$

$$\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_m \subset V,\ c_I^j, b_j \in \mathbb{R}\ , l, u \in \mathbb{R}^V$$

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Multilinear problem: rewriting

$$\min \sum_{I \in \mathcal{I}_0} c_I^0 z_I \tag{7a}$$

$$\text{s.t.} \sum_{I \in \mathcal{I}_j} c_I^j z_I \leq b_j \qquad \forall j \in \{1, \ldots, m\}, \tag{7b}$$

$$z_I = \prod_{v \in I} x_v \qquad \forall I \in \mathcal{E} := \cup_{j=0}^m \mathcal{I}_j, \tag{7c}$$

$$x_v \in [l_v, u_v] \qquad \forall v \in V. \tag{7d}$$

Excursus: multilinear optimization (II)

It is known that there exists an optimal solution in which each x_v is at its bound, that is, $x_v \in \{l_v, u_v\}$ holds for all $v \in V$. Hence, by an affine transformation we can replace by $x_v \in \{0, 1\}$ for all $x_v \in V$.

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The couple $(V, \mathcal{E})$ gives rise to a multilinear polytope defined as the convex hull

$$\mathrm{ML}(V, \mathcal{E}) := \mathrm{conv}\{(x, z) \in \{0,1\}^V \times \{0,1\}^{\mathcal{E}} \mid z_I = \prod_{v \in I} x_v \ \forall I \in \mathcal{E}\} \qquad (8)$$

of the multilinear set defined as the set of the solutions of (7c) with the constraint of x_v being binary.

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of the multilinear set defined as the set of the solutions of (7c) with the constraint of x_v being binary.

As a side note, all of this can be interpreted in the setting of hypergraphs: in particular with $G = (V, \mathcal{E})$ being an hypergraph.

A first relaxation

The simplest polyhedral relaxation of $\mathrm{ML}(V, \mathcal{E})$ is the standard relaxation [SW24]

$$
\begin{aligned}
z_I &\le x_v && \forall v \in I \in \mathcal{E}, && \text{(9a)}\\
z_I + \sum_{v \in I}(1 - x_v) &\ge 1 && I \in \mathcal{E}, && \text{(9b)}\\
z_I &\ge 0 && I \in \mathcal{E}, && \text{(9c)}\\
x_v &\in [0,1] && \forall v \in V. && \text{(9d)}
\end{aligned}
$$

Unfortunately, this relaxation is often very weak and so one could be interested in finding additional constraints to tighten the relaxation.

Extended flower inequalities & Co

One possibility is adding the extended flower inequalities [SW24].

Definition 4.

Let $\mathcal{S} := \{\{v\} \mid v \in V\}$, $I \in \mathcal{E}$, and let $J_1, \ldots, J_k \in \mathcal{E} \cup \mathcal{S}$ be such that $J_1 \cup \cdots \cup J_k \supset I$ and $J_i \cap I \neq \emptyset$ hold for $i = 1, \ldots, k$. The extended flower inequality centered at I with neighbors $J_1, \ldots, J_k$ is the inequality

$$z_I + \sum_{i=1}^{k} (1 - z_{J_i}) \geq 1. \tag{10}$$

Another possibility is the recursive (McCormick) linearization [Kha06].

Affine transformations (I)

A first step could consist in applying the standard relaxation to each monomial in every loop constraint

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A|=2j}} (-1)^j \prod_{h \in A} s_{k_h k_{h+1}} \prod_{l \in A^c} c_{k_l k_{l+1}} = \prod_{i=1}^{n} c_{k_i k_i}.$$

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We have that $c_{k_i k_i} \geq 0$ and, if we suppose that the angle differences are small, we have that $c_{k_l k_{l+1}} \geq 0$. We only need to define variables

$$c'_{k_i k_i} = \frac{c_{k_i k_i}}{V_{k_i}^{\max 2}}, \quad c'_{k_l k_{l+1}} = \frac{c_{k_l k_{l+1}}}{V_{k_l}^{\max} V_{k_{l+1}}^{\max}}, \quad c'_{k_i k_i}, c'_{k_l k_{l+1}} \in [0, 1]. \tag{11}$$

Affine transformations (II)

The problem is that we can not make hypotesis on the sign of $s_{k_h k_{h+1}}$, and rescaling it in the following way

$$s'_{k_h k_{h+1}} = \frac{s_{k_h k_{h+1}} + V^{\max}_{k_h} V^{\max}_{k_{h+1}}}{2 V^{\max}_{k_h} V^{\max}_{k_{h+1}}}$$

would generate a lot more monomials in (5).

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would generate a lot more monomials in (5). The idea is then to write

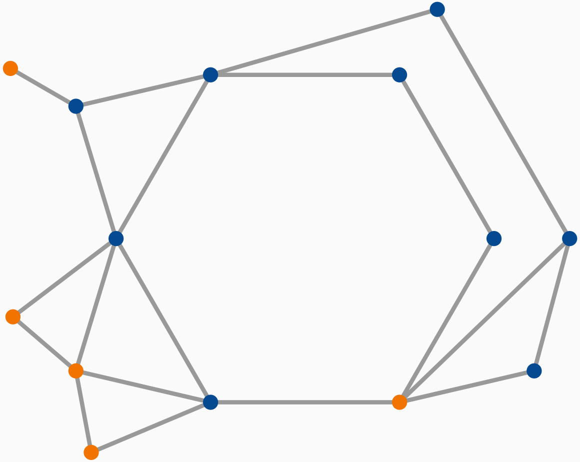
$$s_{k_h k_{h+1}} = V_{k_h}^{\max} V_{k_{h+1}}^{\max} (2\sigma_{k_h k_{h+1}} - 1) u'_{k_h k_{h+1}}, \quad \sigma_{k_h k_{h+1}} \in \{0,1\},\ u_{k_h k_{h+1}} \in [0,1]. \quad (12)$$

At this point $\prod_{ij} s_{ij} = \prod_{ij} V_i^{\max} V_j^{\max} \times \prod_{ij} (2\sigma_{ij} - 1) \times \prod_{ij} u'_{ij}$ and

$$\prod_{ij} (2\sigma_{ij} - 1) = 1 - 2r, \quad \text{where } r + 2m = \sum_{ij} (1 - \sigma_{ij}),\ r \in \{0,1\},\ m \in \mathbb{Z}. \quad (13)$$

Numerical experiment

Preliminary numerical experiment (I)



Network details

- 14 buses, 5 generators, 20 lines;
- minimum cycle basis made up of 7 loops, 5 loops of length 3 and 2 loops of length 6.

Preliminary numerical experiment (II)

	no Jabr	Jabr
no loop	116.090	5.449
loop	84.884	4.717

Table: Norm 2 error of loop constraints (5) with different constraint configurations. Results are scaled by a factor of 10^2.

We studied the impact of the Jabr inequality and our relaxation of the loop constraint on the norm 2 of the vector difference between the lhs and the rhs of (5).

Note that the error on the Jabr equality ($\cong 10^{-9}$) is negligible with respect to the error on the loop constraints when the Jabr inequality is added to the constraints.

Conclusions & future works

Conclusions

- Some models and relaxations for the OPF problem.
- The study of the exactness of the relaxation with respect to the network structure.
- Some multilinear programming techniques and a first attempt to adapt them to the linearization of the loop constraint.

Future works

- Apply the extended flower inequalities or the recursive McCormick linearization to strengthen our formulation.
- Make a comparison between our techniques and the 3-cycle / 4-cycle decomposition.
- Make use of the strong sructure of the loop constraint, investigating the link between multilinear optimization and hypergraph theory [DK17].

Fine.

For other things I do → ambrogiomb.github.io

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