Polynomial Optimization Applied to Power Network Operations

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Workshop on Power Grids

Polynomial Programming

A polynomial program has the following form:

$$\textbf{[PO-P]} \min f(x)$$
$$\text{s.t. } g_i(x) \geq 0 \quad i = \{1, \ldots, m\}$$

In general, solving a polynomial program is $\mathcal{NP}$-hard.

- Relaxations for PO using sums-of-squares decomposition have been shown to be very tight.
  - Sequence of SDP relaxations converging to the optimal.
  - But, computationally expensive to solve in practice.

Research objectives

- Develop new methods for solving general PO.
- Apply these approaches to practical applications.

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A General Recipe for Relaxations of PO

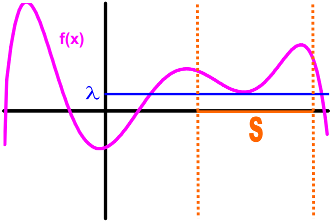
$$\textbf{(PO-P)} \quad z = \min_x \quad f(x) \quad \text{s.t.} \quad g_i(x) \geq 0, \quad i = 1, \ldots, m.$$

(PO-P) is equivalent to

$$\textbf{(PO-D)} \quad \max_\lambda \quad \lambda \quad \text{s.t.} \quad f(x) - \lambda \geq 0 \; \forall x \in S$$

where

$$S := \{x : g_i(x) \geq 0, \; \forall i = 1, \cdots, m\},$$



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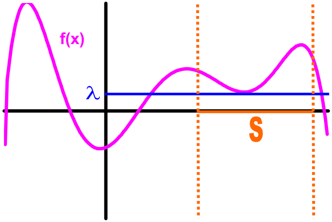
$$S := \{x : g_i(x) \geq 0,\ \forall i = 1, \cdots, m\},$$

$\mathcal{P}_d(S) := \{p(x) \in \mathbf{R}_d[x] : p(x) \geq 0\ \forall x \in S\}$, is the cone of polynomials of degree at most d that are non-negative over S.



A General Recipe for Relaxations of PO

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The condition $f(x) - \lambda \in \mathcal{P}_d(S)$ is $\mathcal{NP}$-hard in general.

We relax it to $f(x) - \lambda \in \mathcal{M}$ for a suitable $\mathcal{M} \subseteq \mathcal{P}_d(S)$.

$$[\text{PO-}\mathcal{M}] \quad \begin{array}{ll} \max & \lambda \\ \text{s.t.} & f(x) - \lambda \in \mathcal{M} \end{array}$$

provides a lower bound for the original problem.

- The choice of $\mathcal{M}$ is a key factor in obtaining good bounds on the problem.

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Sum-of-square Relaxations for PO

[Lasserre 2001, Parillo 2000] For each $r > 0$, define the relaxation,

$$[\text{SOS}_r] \; z_r^{sos} = \quad \max_\lambda \quad \lambda \quad \text{s.t.} \quad f(x) - \lambda \in \mathcal{K}_r$$

provides a lower bound on the original problem where

$$\mathcal{K}_r = SOS_r + \sum_{i=1}^{m} SOS_{r-\deg(g_i)} g_i(x).$$

- For each r, [SOS$_r$] is an SDP program
- As $r \to \infty$, z_r^{sos} converges to global optimum of the original problem.
- As r increases, computational complexity increases rapidly, which makes it impossible to solve large-scale problem in practice.

Overcoming the size blow-up

"Classical" approach:

Use results from algebraic geometry representation to produce hierarchies of approximations converging to the original problem.

Proposed Approach:

- Reduce the problem degree
- Exploit the sparsity characteristics
  - real-world energy networks are represented by sparse graphs where the degree of most nodes in the networks is small
- Develop cheaper relaxations

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Sparse Relaxations of PO

Polynomial Optimization Problem

$$\textbf{[PO-P]}\ z = \min_x \quad f(x) \qquad \text{s.t.} \quad g_i(x) \geq 0, \quad i = 1, \ldots, m.$$

Hierarchy of sparse SDP Relaxations for POP

- [Waki et al. 2006] For each $r > 0$, define the relaxation,

$$[\text{SPSOS}_r]\ z_r^{spsos} = \max_{\lambda, s_{i,k}} \ \lambda \quad \text{s.t.} \quad f(x) - \lambda = \sum_k \left(s_{0,k}(x) + \sum_i s_{i,k}(x) g_i(x) \right), \quad s_{i,k}(x) \text{ is sos supported on } C_k$$

  where C_k is the set of maximal cliques of a chordal extension of the correlative sparsity pattern graph

- as r grows, $z_r^{spsos} \to z$.

Reduction in size

- SDP matrices with size $\binom{|C_k|+r}{r}$ (much smaller than $[\text{SOS}_r]$ if $|C_k| \ll n$).

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SOCP-based Hierarchy for PO

For each $r > 0$, define the relaxation,

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provides a lower bound on the original problem where

$$\mathcal{S}_r = SDSOS_r + \sum_{i=1}^{m} SDSOS_{r-\deg(g_i)} g_i(x)$$

- For each r, $[\text{SDD}_r]$ is a second-order cone program.

Reduction in computational time

- computationally easier to solve
- replacing SOS polynomials with SDSOS polynomials does not guarantee convergence

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Motivation

- In many real-world decision problems, combined challenge of
  - Nonconvex models and dynamics, e.g. energy conservation laws (friction induced headloss, AC power flow).
  - Nonconvex objective functions, e.g. energy costs, risk-averse optimization.
  - Combinations of discrete and continuous decisions, e.g. valve placement, unit commitment, dispatch.
  - Uncertainty in problem parameters, e.g. demands, prices, supply.
- However, we do have
  - Constraints and decision variables are highly structured, e.g. sparsity of traffic, energy or water networks.
  - Samples of realizations for uncertain system parameters; e.g. collected iteratively by sensors and meters
- Optimal decision for these hard, nonconvex real-world problems is in high demand!

Proposed Solution Methods

Decision optimization problem

Mathematical optimization model

↓

Structured Polynomial Optimization Problem

- Add valid inequalities to strengthen convexification.
- Exploit sparsity.
- Efficient algorithms for solving SDPs.
- POP under uncertainty.

↓

Conic relaxations for POP

- Develop new approximation hierarchies.
- Exploit structure in the new conic relaxations.

Energy Networks

Application - AC Optimal Power Flow

Economic dispatch of power generation is a critical problem for utility companies,

Production Cost [O'Neill et al 2012]: $\begin{cases} \text{519\$bn Worldwide} \\ \text{112\$bn USA} \end{cases}$

Goal: Determine the optimal operating point of an electric power generation system.

Challenges:

- non-convex due to the non-linear power flow equations
- lack of global solver for generic power systems

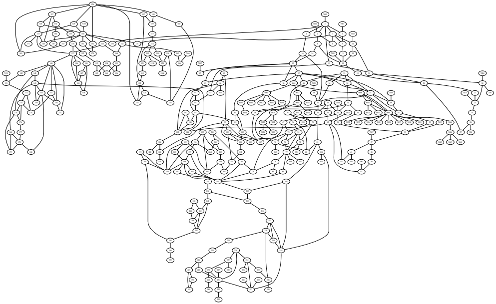
Approach:

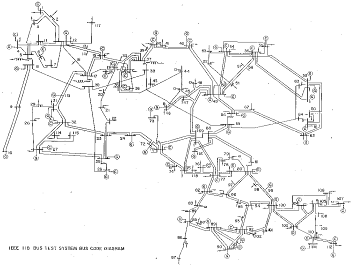
- SDP provide strong bounds for ACOPF [Lavaei & Low 2010]
- Research on polynomial optimization approach.

Benefits:

- Even 1% improvement in dispatch would result in 1-5$bn savings for US (4-20$bn worldwide) [O'Neill et al. 2012]

Challenges - AC Optimal Power Flow





Optimization over power systems:

- Large-scale power transmission and distribution networks.
- AC power flow.
- Integration of distributed, uncertain renewable supply.
- Handling discrete decisions.

ACOPF: Parameters

Sets

N: set of buses	G: set of generators
E: set of branches	L: set of branches with apparent power flow limit

Bus Parameters

$P_k^{\min}$, $P_k^{\max}$:	limits on active generation capacity at bus k.
$Q_k^{\min}$, $Q_k^{\max}$:	limits on reactive generation capacity at bus k.
P_k^d, Q_k^d:	active and reactive load (demand) at each bus k.
$V_k^{\min}$, $V_k^{\max}$:	limits on the absolute value of the voltage at a given bus k.
$y \in \mathbb{R}^{\lvert N\rvert \times \lvert N\rvert}$:	network admittance matrix.

Branch Parameters

$S_{lm}^{\max}$:	limit on the absolute value of the apparent power of a branch (l, m).
$\bar{b}_{lm}$:	total shunt susceptance.
$g_{lm} + jb_{lm}$:	the series admittance of the line.

ACOPF: Parameters

Lavaei and Low notation:

$$
\begin{aligned}
y_k &= e_k e_k^T y, \\
y_{lm} &= (j\frac{\bar{b}_{lm}}{2} + g_{lm} + jb_{lm})e_l e_l^T - (g_{lm} + jb_{lm})e_l e_m^T, \\
Y_k &= \frac{1}{2}\begin{bmatrix} \Re(y_k + y_k^T) & \Im(y_k^T - y_k) \\ \Im(y_k - y_k^T)) & \Re(y_k + y_k^T) \end{bmatrix}, \\
\bar{Y}_k &= -\frac{1}{2}\begin{bmatrix} \Im(y_k + y_k^T) & \Re(y_k - y_k^T) \\ \Re(y_k^T - y_k) & \Im(y_k + y_k^T) \end{bmatrix}, \\
M_k &= \begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix}, \\
Y_{lm} &= \frac{1}{2}\begin{bmatrix} \Re(y_{lm} + y_{lm}^T) & \Im(y_{lm}^T - y_{lm}) \\ \Im(y_{lm} - y_{lm}^T) & \Re(y_{lm} + y_{lm}^T) \end{bmatrix} \\
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\end{aligned}
$$

ACOPF: Formulation

Decision Variables: $x := [\Re V_k \quad \Im V_k]^T$.

$$
\textbf{[OPF-D4]} \quad \begin{aligned} \min\ & \text{Power Generation Cost} \\ \text{s.t.}\ & \text{Active Power Constraint} \\ & \text{Reactive Power Constraint} \\ & \text{Voltage Constraint} \\ & \text{Apparent Power Flow Constraint} \end{aligned}
$$

ACOPF: Formulation

Decision Variables: $x := [\Re V_k \quad \Im V_k]^T$.

$$\textbf{[OPF-D4]} \quad \min \sum_{k \in G} \left(c_k^2 (P_k^d + \mathrm{tr}(Y_k x x^T))^2 + c_k^1 (P_k^d + \mathrm{tr}(Y_k x x^T)) + c_k^0 \right)$$

s.t. Active Power Constraint

Reactive Power Constraint

Voltage Constraint

Apparent Power Flow Constraint

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$$\text{s.t.} \quad P_k^{\min} \le \mathrm{tr}(Y_k x x^T) + P_k^d \le P_k^{\max}$$

Reactive Power Constraint

Voltage Constraint

Apparent Power Flow Constraint

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$$\text{s.t. } P_k^{\min} \leq \mathrm{tr}(Y_k x x^T) + P_k^d \leq P_k^{\max}$$

$$Q_k^{\min} \leq \mathrm{tr}(\bar{Y}_k x x^T) + Q_k^d \leq Q_k^{\max}$$

Voltage Constraint

Apparent Power Flow Constraint

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$$Q_k^{\min} \leq \mathrm{tr}(\bar{Y}_k x x^T) + Q_k^d \leq Q_k^{\max}$$

$$(V_k^{\min})^2 \leq \mathrm{tr}(M_k x x^T) \leq (V_k^{\max})^2$$

Apparent Power Flow Constraint

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$$\begin{aligned} \text{s.t.} \quad & P_k^{\min} \le \mathrm{tr}(Y_k x x^T) + P_k^d \le P_k^{\max} \\ & Q_k^{\min} \le \mathrm{tr}(\bar{Y}_k x x^T) + Q_k^d \le Q_k^{\max} \\ & (V_k^{\min})^2 \le \mathrm{tr}(M_k x x^T) \le (V_k^{\max})^2 \\ & (\mathrm{tr}(Y_{lm} x x^T))^2 + (\mathrm{tr}(\bar{Y}_{lm} x x^T))^2 \le (S_{lm}^{\max})^2 \end{aligned}$$

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\text{s.t.} \ & P_k^{\min} \le \mathrm{tr}(Y_k x x^T) + P_k^d \le P_k^{\max} \\
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& (\mathrm{tr}(Y_{lm} x x^T))^2 + (\mathrm{tr}(\bar{Y}_{lm} x x^T))^2 \le (S_{lm}^{\max})^2
\end{aligned}
$$

Optimal Power Flow is a polynomial optimization problem of degree 4, hard to solve to optimality even for small instances [Molzahn and Hiskens 2013][Josz et al. 2013].

ACOPF: Quadratic PO

$$
\begin{aligned}
\textbf{[OPF-Q]} \quad \min \sum_{k \in G} & \left(c_k^2 (P_k^g)^2 + c_k^1 (P_k^d + \mathrm{tr}(Y_k x x^T)) + c_k^0 \right) \\
& P_k^{\min} \leq \mathrm{tr}(Y_k x x^T) + P_k^d \leq P_k^{\max} \\
& Q_k^{\min} \leq \mathrm{tr}(\bar{Y}_k x x^T) + Q_k^d \leq Q_k^{\max} \\
& (V_k^{\min})^2 \leq \mathrm{tr}(M_k x x^T) \leq (V_k^{\max})^2 \\
& P_{lm}^2 + Q_{lm}^2 \leq (S_{lm}^{\max})^2 \\
& P_k^g = \mathrm{tr}(Y_k x x^T) + P_k^d \\
& P_{lm} = \mathrm{tr}(Y_{lm} x x^T) \\
& Q_{lm} = \mathrm{tr}(\bar{Y}_{lm} x x^T)
\end{aligned}
$$

[OPF-Q] has $|G| + 2|L|$ additional variables which can be relatively small as $|G| \ll |N|$ and $|L| \ll |E|$.

Optimal Power Flow - Duality

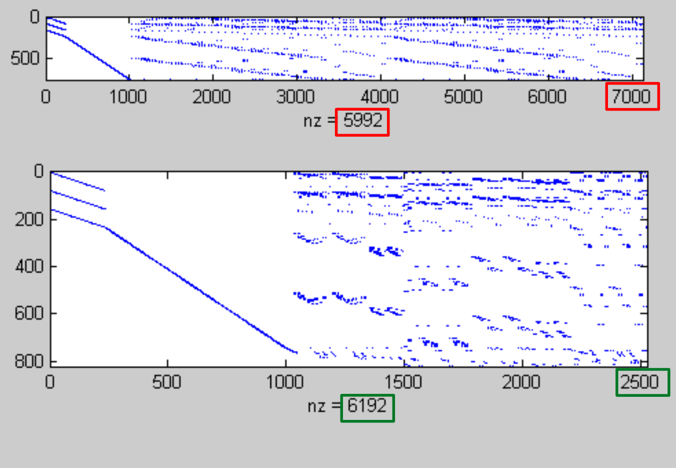
Using duality, the following results hold for the ACOPF problem:

1. The first level of the [SOS$_r$] hierarchy of [OPF-Q] is the conic dual of **Optimization 3** Proposed by Lavaei and Low, 2010.
2. The first level of the [SDD$_r$] hierarchy of [OPF-Q] is the conic dual of **Problem** $\mathcal{R}_2$ Proposed by Low 2013.

Exploiting Sparsity

- Sparsity of admittance matrix can be exploited. [Stott 1974].
- Exploit sparsity in SDP relaxation for OPF [Molzahn et al. 2013].
- SparseCoLO package [Fujisawa et al. 2010], [Kim et al. 2010].

Sparsity of the SDP relaxation: 39 Buses



	n	nnz(A)	sum(SDP_size)	max(SDP_size)	#SDP Blocks	CPU$_t$
SDP	7114	5992	6880	78	95	6.4
S-SDP	2526	6192	2292	18	103	0.3

OPF - Results

Test Case	$\|N\|$	$\|E\|$	Gap (%) [SDD$_2$]	Gap (%) [SOS$_2$]	Gap (%) [SPSOS$_2$]	Runtime (seconds) [SDD$_2$]	Runtime (seconds) [SOS$_2$]	Runtime (seconds) [SPSOS$_2$]
case3_lmbd	3	3	1.32	0.39	0.39	<1	<1	<1
case5_pjm	5	6	14.55	5.22	5.22	<1	<1	<1
case14_ieee	14	20	0.11	0	0	<1	<1	<1
case24_ieee_rts	24	38	0.02	0	0	<1	<1	<1
case30_as	30	41	0.06	0	0	<1	2	<1
case30_fsr	30	41	0.39	0	0	<1	2	<1
case30_ieee	30	41	10.81	0.01	0.01	<1	2	<1
case39_epri	39	46	0.49	0.01	0.01	<1	6	<1
case57_ieee	57	80	0.46	0.01	0	<1	20	<1
case73_ieee_rts	73	120	0.04	0	0	<1	60	<1
case89_pegase	89	210	0.75	0.01	0.01	<1	160	<1
case118_ieee	118	186	2.27	0.18	0.18	<1	608	<1
case162_ieee_dtc	162	284	7.68	n.d.	2.26	<1	n.d.	3
case179_goc	179	263	0.13	n.d.	0.06	<1	n.d.	<1
case200_tamu	200	245	0.01	n.d.	0	<1	n.d.	<1
case240_pserc	240	448	3.92	n.d.	2.28	<1	n.d.	2
case300_ieee	300	411	2.6	n.d.	0.11	<1	n.d.	2
case500_tamu	500	597	5.39	n.d.	2.11	<1	n.d.	2
case588_sdet	588	686	2.10	n.d.	0.67	<1	n.d.	3
case1354_pegase	1354	1991	2.44	n.d.	0.56	3	n.d.	7
case1888_rte	1888	2531	2.06	n.d.	1.75	4	n.d.	11
case1951_rte	1951	2596	0.50	n.d.	0.02	5	n.d.	11
case2000_tamu	2000	3206	0.21	n.d.	–	3	n.d.	119
case2316_sdet	2316	3017	2.30	n.d.	0.73	9	n.d.	141
case2383wp_k	2383	2896	1.21	n.d.	0.38	4	n.d.	80
case2736sp_k	2736	3504	2.35	n.d.	0.01	4	n.d.	75
case2737sop_k	2737	3506	11.13	n.d.	0.02	2	n.d.	69
case2746wop_k	2746	3514	2.01	n.d.	0.01	4	n.d.	91
case2746wp_k	2746	3514	18.24	n.d.	0.01	3	n.d.	90

OPF - Results

Test Case	$\|N\|$	$\|E\|$	Gap (%) [SDD$_2$]	Gap (%) [SOS$_2$]	Gap (%) [SPSOS$_2$]	Runtime (seconds) [SDD$_2$]	Runtime (seconds) [SOS$_2$]	Runtime (seconds) [SPSOS$_2$]
case2848_rte	2848	3776	0.41	n.d.	0.05	6	n.d.	21
case2853_sdet	2853	3921	3.09	n.d.	0.55	8	n.d.	61
case2868_rte	2868	3808	0.55	n.d.	0.21	7	n.d.	19
case2869_pegase	2869	4582	1.08	n.d.	0.42	9	n.d.	26
case3012wp_k	3012	3572	15.28	n.d.	0.17	4	n.d.	127
case3120sp_k	3120	3693	15.61	n.d.	0.14	4	n.d.	139
case3375wp_k	3375	4161	1.60	n.d.	n.d.	5	n.d.	n.d.
case4661_sdet	4661	5997	10.24	n.d.	n.d.	21	n.d.	n.d.
case6468_rte	6468	9000	2.56	n.d.	0.47	18	n.d.	174
case6470_rte	6470	9005	3.88	n.d.	0.47	23	n.d.	210
case6495_rte	6495	9019	18.07	n.d.	14.76	25	n.d.	232
case6515_rte	6515	9037	8.52	n.d.	6.46	24	n.d.	214
case9241_pegase	9241	16049	2.94	n.d.	2.18	65	n.d.	524
case10000_tamu	10000	12706	0.82	n.d.	0.39	21	n.d.	1009
case13659_pegase	13659	20467	1.66	n.d.	n.d.	74	n.d.	n.d.

Paper: Optimal Power Flow as a Polynomial Optimization Problem, IEEE Transactions on Power Systems.
Paper: Alternative LP and SOCP Hierarchies for ACOPF problems, IEEE Transactions on Power Systems.

Current Work

Current Work

Uncertainty

- develop methodologies to handle uncertainty
- apply to practical problems (ACOPF with uncertain demand)

Conic relaxations

- combine SOCP and SDP relaxations
- apply to MIQCQP (multiperiod ACOPF with binary variables)

Uncertainty in Power Systems

ACOPF with uncertainty

- Demand uncertainty and renewable energy penetration
- Adjustable Robust QCQP with ellipsoidal uncertainty

Paper: Adjustable Robust Two-Stage Polynomial Optimization with Application to AC Optimal Power Flow, SIAM Journal on Optimization, 2023.

ACOPF as a non-convex quadratic optimization problem

$$
\begin{aligned}
[QP]: \quad & \min_{y,x} \quad y^{\mathrm{T}} P y + p^{\mathrm{T}} y + p_0 \quad \text{(convex)} \\
& \text{s.t.} \quad Ay \leq b \quad \text{(convex)} \\
& \qquad x^{\mathrm{T}} Q_i x + q_i = y_i \ \text{ for all } i \in \{1, \ldots, m_{eq}\} \quad \text{(non-convex)} \\
& \qquad x^{\mathrm{T}} Q_j x + q_j \geq 0 \ \text{ for all } j \in \{1, \ldots, m_{in}\} \quad \text{(non-convex)}
\end{aligned}
$$

- y are control variables (e.g., active power on PV buses)
- x are state variables (voltages in rectangular form)

Adjustable Robust ACOPF

- Consider uncertainty in power demands or generation
- Ellipsoidal uncertainty set: $\Omega = \{\zeta \in \mathbb{R}^{n_\zeta} : \zeta^{\mathrm{T}}\Sigma\zeta \leq 1\}$

$$
\begin{aligned}
[ARQP]: \quad \min_{y} \quad & y^{\mathrm{T}}Py + p^{\mathrm{T}}y + p_0 \\
\text{s.t.} \quad & Ay \leq b \\
& \text{and for any } \zeta \in \Omega \text{ there is } x \text{ such that:} \\
& x^{\mathrm{T}}Q_i x + m_i^{\mathrm{T}}\zeta + q_i = y_i \quad \text{for all } i \in \{1, \ldots, m_{eq}\} \\
& x^{\mathrm{T}}Q_j x + m_j^{\mathrm{T}}\zeta + q_j \geq 0 \quad \text{for all } j \in \{1, \ldots, m_{in}\}
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- How to approach “robust” equalities?

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- How to approach "robust" equalities?

Affine equalities

Eliminate x, obtain a robust problem in y

$$\begin{aligned} \min_{y} \quad & y^{\mathrm{T}} P y + p^{\mathrm{T}} y + p_0 \\ \text{s.t.} \quad & Ay \leq b \\ & \text{and for any } \zeta \in \Omega \text{ there is } x \text{ such that:} \\ & D_1 \zeta + D_2 y + d = x \\ & x^{\mathrm{T}} Q_j x + m_j^{\mathrm{T}} \zeta + q_j \geq 0 \quad \text{for all } j \in \{1, \ldots, m_{in}\} \end{aligned}$$

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\text{s.t.} \quad & Ay \leq b \\
& \text{and for any } \zeta \in \Omega,\ j \in \{1, \ldots, m_{in}\} \\
& (D_1 \zeta + D_2 y + d)^{\mathrm{T}} Q_j (D_1 \zeta + D_2 y + d) + m_j^{\mathrm{T}} \zeta + q_j \geq 0
\end{aligned}$$

Affine equalities

Eliminate ζ, obtain an SDP in y

$$
\begin{aligned}
= \quad \min_{y} \quad & y^{\mathrm{T}} P y + p^{\mathrm{T}} y + p_0 \\
\text{s.t.} \quad & Ay \leq b \\
& \text{and for all } \zeta^{\mathrm{T}} \Sigma \zeta \leq 1,\ j \in \{1, \dots, m_{in}\} \\
& \zeta^{\mathrm{T}} A_j \zeta + (y^{\mathrm{T}} B_j + b_j^{\mathrm{T}}) \zeta + (y^{\mathrm{T}} C_j + c_j^{\mathrm{T}}) y + d_j \geq 0
\end{aligned}
$$

Affine equalities

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\begin{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
\overset{\text{S-lemma}}{=} \quad & \min_{y, \lambda, \gamma} \quad y^{\mathrm{T}} P y + p^{\mathrm{T}} y + p_0 \qquad \text{(convex, tractable)} \\
& \text{s.t.} \quad Ay \leq b \qquad \text{(convex, tractable)} \\
& \qquad \text{and for all } j \in \{1, \ldots, m_{in}\} \\
& \qquad \begin{bmatrix} \gamma_j + c_j^{\mathrm{T}} y + d_j - \lambda_j, & \frac{1}{2}(y^{\mathrm{T}} B_j + b_j^{\mathrm{T}}) \\ \frac{1}{2}(B_j^{\mathrm{T}} y + b_j), & \lambda_j \Sigma + A_j \end{bmatrix} \succeq 0 \qquad \text{(convex, tractable)} \\
& \qquad \lambda_j \geq 0 \qquad \text{(convex, tractable)} \\
& \qquad y^{\mathrm{T}} C y = \gamma_j \qquad \textbf{(non-convex, but doable)}
\end{aligned}
$$

Algorithm to solve [ARQP]

- Approximate quadratic in x (state var.) functions in each equality by piecewise affine functions in x
- Express x as a function of (y, ζ), eliminate x and equalities
  Result: quadratic *robust* optimization problem in (y, ζ)
- Use S-lemma to eliminate ζ (uncertainty var.)
  Result: SDP in y (control var.) with quadratic equalities
- Solve SDP with quadratic equalities via Alternating Projections
  Result: robust control var. solution to the piecewise affine approximation of [ARQP]
- Check feasibility of the above control var. solution for [ARQP]
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Details of piecewise-affine approximations

- Partition the feasible set for x into subsets $S_1, \ldots, S_J$
- Apply "Algorithm to solve [ARQP]" on restrictions of [ARQP] to each subset $S_k, k \leq J$. Use affine approximations on S_k:
  - Let $\hat{x}$ be the "center" of S_k
  - Linearize equality constraints using Taylor series:

    $x^{\mathrm{T}} Q_i x \to \hat{x}^{\mathrm{T}} Q_i \hat{x} + \hat{x}^{\mathrm{T}} Q_i (x - \hat{x})$ for all $i \in \{1, \ldots, m_{eq}\}$
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Numerical experiments: instances with up to 9 buses

Uncertainty, % of load	Model	Average objective	Time, sec.	Constraint violations, %	Max # violations, per exper., PQ	Max # violations, per exper., VI
case 6ww, 6 buses						
1	Nominal	31.3	0.0	43.9	0	2
	DCOPF	-	-	-	-	-
	SDP	31.4	28.4	22.1	0	2
	Taylor	31.6	69.4	**0.0**	0	0
case 9, 9 buses						
1	Nominal	53.0	0.0	0.7	0	2
	DCOPF	53.2	9.1	**0.0**	0	0
	SDP	53.0	22.5	100.0	0	2
	Taylor	53.3	57.6	**0.0**	0	0
5	Nominal	53.2	0.0	35.7	0	2
	DCOPF	53.5	13.8	**0.0**	0	0
	SDP	53.2	30.5	100.0	0	3
	Taylor	53.4	72.7	**0.0**	0	0
10	Nominal	53.5	0.0	43.9	0	2
	DCOPF	54.4	12.6	0.1	1	0
	SDP	53.6	23.8	87.5	0	3
	Taylor	53.7	70.4	**0.0**	0	0
20	Nominal	54.9	0.0	48.5	1	5
	DCOPF	55.5	12.1	3.4	1	0
	SDP	55.0	29.4	99.8	1	6
	Taylor	55.0	74.4	**1.0**	1	0
30	Nominal	57.1	0.0	51.3	1	6
	DCOPF	-	-	-	-	-
	SDP	57.4	26.1	97.8	1	6
	Taylor	57.2	68.1	**7.1**	1	2

Numerical experiments: instances with 30 to 118 buses

Uncertainty, % of load	Model	Average objective	Time, sec.	Constraint violations, %	Max # violations, per exper., PQ	Max # violations, per exper., VI
case 30, 30 buses						
1	Nominal	6.1	0.1	100.0	0	2
	DCOPF	-	-	-	-	-
	SDP	5.8	178.6	31.7	0	2
	Taylor	5.9	177.2	**1.7**	0	2
case 57, 57 buses						
1	Nominal	417.4	0.1	70.6	2	1
	DCOPF	418.5	43.0	100.0	2	1
	Taylor	426.8	467.2	**0.0**	0	0
case 118, 118 buses						
1	Nominal	1296.7	0.2	99.5	9	0
	DCOPF	1315.6	94.9	100.0	21	0
	Taylor	1301.3	830.0	**1.1**	1	0

Conclusions

- Solved large-scale deterministic problems
- Solved small and medium scale problems with uncertainty
  - Finds control solutions in short time for small to middle-sized MATPOWER cases
  - Works best for low to moderate levels of uncertainty
  - Generalizes to adjustable robust polynomial problems
- **Ongoing:** Solve small and medium scale problems with unit commitment and AC constraints
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Thank you!